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13. SUPPLEMENTARY NOTES
The main accomplishments are the development of 1) an algorithm for determining the shape and surface conductivity of a partially coated dielectric, 2) the reciprocity gap functional method for imaging of subsurface caves and tunnels and 3) a formula for determining a lower bound for the index of refraction, all from a knowledge of the scattered field.

14. ABSTRACT

This grant is concerned with the problem of the detection of decoys and objects buried in the earth from measured electromagnetic scattering data at fixed frequency in the resonance region. The proposed research includes the problem of determining the shapes and material properties of the unknown scattering objects, in particular the surface conductivity and index of refraction. Central to this investigation has been the study of a new class of elliptic problems called the interior transmission problems as well as the use of the ultra-weak method to solve direct scattering problems in order to generate data for the inverse problem.

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Final Performance Report

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Objectives

This grant is concerned with the problem of the detection of decoys and objects buried in the earth from measured electromagnetic scattering data at fixed frequency in the resonance region. The proposed research includes the problem of determining the shapes and material properties of the unknown scattering objects, in particular the surface conductivity and index of refraction. Central to this investigation has been the study of a new class of elliptic problems called the interior transmission problems as well as the use of the ultra-weak method to solve direct scattering problems in order to generate data for the inverse problem.

Status of Effort

During the period of this report particular attention was paid to scattering by a dielectric that is partially coated by a thin highly conducting layer [7], [8], [10], [14], [19]. Such problems arise when a benign object is coated in an effort to make it appear hostile to radar interrogation. Since in general the coating does not cover the entire object, the direct scattering problem in such situations leads to the investigation of a mixed boundary value problem for Maxwell's equations and such an investigation is only in its infancy. The corresponding inverse problem is rendered particularly difficult since neither the nature nor the extent of the coating is known a priori. Using the linear sampling method, we have developed a numerical method for solving the above inverse scattering problem for a partially coated dielectric. In particular, we have developed an algorithm for determining the shape and surface conductivity of a partially coated dielectric from a knowledge of the electric far field patterns corresponding to time harmonic incident plane waves at fixed frequency. No a priori assumption is made on the material properties of the dielectric, the connectivity of the scattering obstacle nor on the extent of the coating.

A second problem that we have investigated is the imaging of subsurface caves and tunnels using multistatic electromagnetic scattering data [6], [12], [14]. Conventional radar imaging techniques use backscattered data to image targets and inversion algorithms are mainly based on the use of weak scattering approximations such as the Born or Kirchhoff approximations. Such an approach leads to very simple linear models but at the expense of ignoring multiple scattering and polarization effects. Although highly successful in certain applications, the use of backscattered data makes successful 3-D imagining in complicated environments extremely problematic. In view of the current ability to collect greater amounts of data, during the period of this report we have developed a new method for subsurface imagining called the *reciprocity gap functional method* which avoids the need to rely on "weak scattering" methods and at the same time makes use of multistatic data and polarization effects. This method is particularly attractive since 1) even though it avoids the the need for the "weak scattering" approximation it is still a linear method that is easily implementable and 2) it is not necessary to make a priori assumptions on the material or geometric properties of the objects being imaged. The second point is of particular importance in order to minimize false negative assurances and false positive alarms. A third attractive feature of our method is that it avoids the need to compute the Green's function for the background medium.

A third problem that we have considered during the period of this report is the interior transmission problem [9], [13], [20]. The interior transmission problem is a boundary value problem arising in inverse scattering theory that, to our knowledge, is not covered by any existing theory in partial differential equations. Nevertheless, the problem is easy to state and a better understanding of conditions under which the problem is well posed would almost surely lead to major advances in inverse scattering theory. Of particular importance is the spectral theory associated with the class of boundary value problems of which essentially nothing is known in more then one dimension. In this direction we have shown that for a dielectric isotropic scatterer having index of refraction greater than one then the index of refraction is bounded below by $\lambda(D)/k_0^2$ where $\lambda(D)$ is the first Dirichlet eigenvalue for the Laplacian in D, D is the scattering obstacle and k_0 is the first transmission eigenvalue. These results have also been partiallyextended to the case of Maxwell's equations. Since D and k_0 can be determined from the far field data, this then provides a lower bound for the index of refraction from the measured data. Research in this direction is an ongoing project.

We have also participated in discussions with Dr. Albanese (Brooks AFB, San Antonio) on the inverse problem of detecting the source of electrical activity in the brain from measurements on the skull. This has resulted in a paper [2] on uniqueness questions for this problem.

Central to all of the above investigations is the ability to numerically solve the direct scattering problem in order to generate data for testing our inversion algorithms. This has been done through the further development of the ultra-weak method (a special discontinuous Galerkin method) for the numerical solution of electromagnetic scattering problems [1], [21], [22], [23]. In addition, progress has been made on the development of variational methods for rough surface scattering problems such as may arise in subsurface imaging [16], [17].

An introduction to our approach to inverse scattering theory has recently appeared in the monograph [5].

Accomplishments/New Findings

The main accomplishments during the period of this report were:

1. The development of an algorithm for determining the shape and surface conductivity of a partially coated dielectric from a knowledge of the electric far field patterns corresponding to time harmonic incident plane waves.

2. The development of the reciprocity gap functional method for imaging of subsurface caves and tunnels using multistatic electromagnetic scattering data.

3. The derivation of a formula for determining a lower bound for the index of refraction from a knowledge of the far field pattern of the scattered electromagnetic wave.

Personnel Supported

1. Faculty
   D. Colton, P. Monk and F. Cakoni (Principal Investigators)

2. Short- Term Visitors
   Tilo Arens, University of Karlsruhe, Germany
   Andreas Kirsch, University of Karlsruhe, Germany
   Lassi Päivärinta, University of Helsinki, Finland
   Eric Darrigrand, University of Rennes, France
   Houssem Haddar, INRIA, France
   Rainer Kress, University of Göttingen, Germany
   Tomi Huttunen, University of Kuopio, Finland
   Virginia Selgas, University of Coruña, Spain.

Interactions/Transitions

Professors Colton, Monk and Cakoni have attended numerous conferences and seminars as invited speakers both in this country and in Europe.

Publications (with abstract)

1. M. Ainsworth, P.B. Monk and W. Muniz, Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second order wave equation, *Jour. Scientific Computing* **27**, 5-40 (2006).

   Abstract: Discontinuous Galerkin finite element methods (DGFEM) offer certain advantages over standard continuous finite element methods when applied to the spatial discretisation of the acoustic wave equation. For instance, the mass matrix has a block diagonal structure which, used in conjunction with an explicit time stepping scheme, gives an extremely economical scheme for time domain simulation. This feature is ubiquitous and extends to other time dependent wave problems such as Maxwell's equations. An important considerationin computational wave propagation is the dispersive and dissipative properties of the discretisation scheme in comparison with those of the original system. We investigate these properties for two popular DGFEM schemes: the interior penalty discontinuous Galerkin finite element method applied to the second order wave equation and a more general family of schemes applied to the corresponding first order system. We show how the analysis of the multi-dimensional case may be reduced to consideration of one dimensional problems. We derive the dispersion error for various schemes and conjecture on the generalization to higher order approximations in space.

2. R. Albanese and P.B. Monk, The inverse source problem for Maxwell's equations, *Inverse Problems* **22**, 1023-1035 (2006).

   Abstract: The inverse source problem for Maxwell's equations is considered. We show that the problem of finding a volume current density from surface measurements does not have a unique solution and characterize the non-uniqueness. We also show that if further a priori information is available, the inverse source problem may have a unique solution (in particular for surface currents or dipole sources).

3. F. Cakoni, Recent developments in the qualitative approach to inverse electromagnetic scattering theory, *J. Comput. Appl. Math.* **204**, 242-255 (2007).

   Abstract: We consider the inverse scattering problem of determining both the shape and some of the physical properties of the scattering object from a knowledge of the (measured) electric and magnetic fields due to the scattering of an incident time harmonic electromagnetic wave at fixed frequency. We shall discuss the *linear sampling method* for solving the inverse scattering problem which does not require any a priori knowledge of the geometry and the physical properties of the scatterer. Included in our discussion is the case of partially coated objects and inhomogeneous background. We give references for numerical examples for each problem discussed in this paper.

4. F. Cakoni and D. Colton, Open problems in the qualitative approach to inverse electromagnetic scattering theory, *European Jour. Appl. Math.* **16**, 1-15 (2005).

   Abstract We formulate a number of open problems for time-harmonic inverse electromagnetic scattering theory focusing on uniqueness theorems, the determination of the support of a scattering object and the determination of material parameters

5. F.Cakoni and D.Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Series on Interaction of Mechanics and Mathematics (2006).

   Abstract: This is a book. Inverse scattering theory has been a particularly active and successful field in applied mathematics and engineering for the past twenty years. The increasing demands of imaging and target identification require new techniques besides the existing weak scattering approximation or nonlinear optimization methods. One class of such methods comes under the general description of qualitative methods in inverse scattering theory. This textbook is an easily-accessible "class-tested" introduction to the field. It is accessible also to readers who are not professional mathematicians, thus making these new mathematical ideas in inverse scattering theory available to the wider scientific and engineering community.

6. F. Cakoni and D. Colton, Target identification of buried coated objects, *Comput. Appl. Math.* **25**, 269-288 (2006).

   Abstract: We consider the three dimensional electromagnetic inverse scattering problem of determining information about a buried coated object from a knowledge of the electric and magnetic fields measured on the surface of the earth corresponding to time harmonic electric dipoles as incident fields. We assume that the buried object is a perfect conductor that is (possibly) partially coated by a thin dielectric layer. No a priori assumption is made on the extent of the coating, i.e. the object can be fully coated, partially coated or not coated at all. We present an algorithm based on the linear sampling method and reciprocity gap functional for reconstructing the shape of the scattering obstacle together with an estimate of the surface impedance of the coating.

7. F. Cakoni and D. Colton, Inequalities in inverse scattering theory, *J. Inverse and Ill-Posed Problems* **15**, 483-491 (2007).

   Abstract: We consider the scattering of time harmonic electromagnetic plane waves by a bounded, inhomogeneous dielectric medium that is partially coated by a thin metallic layer in $\mathbb{R}^2$. We use the far field pattern of the scattered waves at a fixed frequency as data to determine the support D of the inhomogeneous obstacle, the surface conductivity and the relative permittivity. No a prior information on the material properties of the scatterer is needed. The support D is determined by the linear sampling method which is based on an approximate solution of the far field equation. This solution is also used to obtain lower bounds for the surface conductivity and relative permittivity. The techniques for solving this inverse scattering problem rely on the analysis of a non standard boundary value problem known as the interior transmission problem.

8. F. Cakoni, D. Colton and P. Monk, The determination of the surface conductivity of a partially coated dielectric, *SIAM J. Appl. Math.* **65**, 767-789 (2005).

   Abstract: A variational method is given for determining the essential supremum of the surface conductivity of a partially coated anisotropic dielectric medium from a knowledge of the far field pattern of the time-harmonic electric field at fixed frequency corresponding to an incident plane wave. It is assumed that the shape of the scatterer has been determined (e.g. by solving the far field equation and using the linear sampling method). Numerical examples are given for the scalar case with constant surface conductivity.

9. F. Cakoni, D. Colton and P. Monk, On the use of transmission eigenvalues to estimate the index of refraction from far filed data, *Inverse Problems* **23**, 507-522 (2007).

   Abstract: We consider the scattering of time harmonic electromagnetic plane waves by a bounded inhomogeneous medium and show that under certain assumptions a lower bound on the index of refraction can be obtained from a knowledge of the smallest transmission eigenvalue corresponding to the medium. It is then shown by numerical examples that this eigenvalue can be determined from a knowledge of the far field pattern of the scattered wave, thus providing a practical method for estimating the index of refraction from far field data.

10. F. Cakoni, D. Colton and P. Monk, The inverse electromagnetic scattering problem for a partially coated dielectric, *J. Comput. Appl. Math.* **204**, 256-267 (2007).

    Abstract: We use the linear sampling method to determine the shape and surface conductivity of a partially coated dielectric infinite cylinder from knowledge of the far field pattern of the scattered TM polarized electromagnetic wave at fixed frequency. A mathematical justification of the method is provided based on the use of a complete family of solutions. Numerical examples are given showing the efficiency of our method.

11. F. Cakoni and E. Darrigrand, The inverse electromagnetic scattering problem for a mixed boundary value problem for screens, *J. Comput. Appl. Math.* **174**, 251-269 (2005).

    Abstract: We consider the inverse scattering problem of determining the shape of mixed perfectly conducting-impedance screens from a knowledge of the incident time harmonic electromagnetic plane wave and the electric far field pattern of the scattered wave. We adapt the linear sampling method invented by Colton and Kirsch (1996 *Inverse Problems*, 12, 383-393) for the case of scattering by obstacles with nonempty interior. Numerical examples are given for mixed screens in $\mathbb{R}^3$.

12. F. Cakoni, M.B. Fares and H. Haddar, Analysis of two linear sampling methods applied to electromagnetic imagining of buried objects, *Inverse Problems* **22**, 845-867 (2006) .

    Abstract: We investigate two imaging methods to detect buried scatterers from electromagnetic measurements at fixed frequency. The first one is the classical linear sampling method that requires the computation of the Green's tensor for the background medium. This job can be numerically very costly for complex background geometries. The second one is an alternative approach based on the reciprocity gap functional that avoids the computation of the Green's tensor but requires the knowledge of both the electric and magnetic fields. Numerical examples are given showing the performance of both methods.

13. F. Cakoni and H. Haddar, A variational approach for the solution of electromagnetic interior transmission problem for anisotropic media, *Inverse Problems and Imaging* **1**, 443-456 (2007).

    Abstract: The interior transmission problem plays a basic role in the study of inverse scattering problems for inhomogeneous media. In this paper we study the interior transmission problem for the Maxwell equations in the electromagnetic scattering problem for an anisotropic inhomogeneous object using a variational approach. In addition, we also describe the structure of the transmission eigenvalues.

14. F. Cakoni and H. Haddar, Identification of partially coated anisotropic buried objects using electromagnetic Cauchy data, *J. Integral Equations and Applications* **19**, 361-391 (2007).

    Abstract: We consider the three dimensional electromagnetic inverse scattering problem of determining information about a target buried in a known inhomogeneous medium from a knowledge of the electric and magnetic fields corresponding to time harmonic electric dipoles as incident fields. The scattering object is assumed to be an anisotropic dielectric that is (possibly) partially coated by a thin layer of highly conducting material. The data is measured at a given surface containing the object in its interior. Our concern is to determine the shape of this scattering object and some information on the surface conductivity of the coating without any knowledge of the index of refraction of the inhomogeneity. No a priori assumption is made on the extent of the coating, i.e. the object can be fully coated, partially coated or not coated at all. Our method is based on the linear sampling method and reciprocity gap functional for reconstructing the shape of the scattering object. The algorithm consists in solving a set of linear integral equations of the first kind for several sampling points and three linearly independent polarizations. The solution of these integral equations is also used to determine the surface conductivity.

15. F. Cakoni and R. Kress, Integral equations for inverse problems in corrosion detection from partial Cauchy data, *Inverse Problems and Imagining* **1**, 299-245 (2007).

    Abstract: We consider the inverse problem to recover a part Γ_c of the boundary of a simply connected planar domain D from a pair of Cauchy data of a harmonic function u in D on the remaining part $\partial D \setminus \Gamma_c$ when u satisfies a homogeneous impedance boundary condition on Γ_c. Our approach extends a method that has been suggested by Kress and Rundell for recovering the interior boundary curve of a doubly connected planar domain from a pair of Cauchy data on the exterior boundary curve and is based on a system of nonlinear integral equations. As a byproduct, these integral equations can also be used to extend incomplete Cauchy data and to solve the inverse problem to recover an impedance profile on a known boundary curve. We present the mathematical foundation of the method and illustrate its feasibility by numerical examples.

16. S.N. Chandler-Wilde and P.B. Monk, Existence, uniqueness and variational methods for scattering by unbounded rough surfaces, *SIAM Jour. Math. Analysis* **37**, 598-618 (2005).

    Abstract: In this paper we study, via variational methods, the problem of scattering of time harmonic acoustic waves by an unbounded sound soft surface. The boundary ∂D is assumed to lie within a finite distance of a flat plane and the incident wave is that arising from an inhomogeneous term in the Helmholtz equation whose support lies within some finite distance of the boundary ∂D. Via analysis of an equivalent variational formulation, we provide the first proof of existence of a unique solution to a 3D rough surface scattering problem for arbitrary wave number. Our method of analysis does not require any smoothness of the boundary which can, for example, be the graph of an arbitrary bounded continuous function. An attractive feature is that all constants in a priori bounds, for example the inf-sup constant of the sesquilinear forms, are bounded by explicit functions of the wave number and the maximum surface elevation.

17. S.N. Chandler-Wilde, P.B. Monk and M. Thomas, The mathematics of scattering by unbounded, rough, inhomogeneous layers, *J. Comp. Appl. Math.* **204**, 249-259 (2007).

    Abstract: In this paper we study, via variational methods, a boundary value problem for the Helmholtz equation modelling scattering of time harmonic waves by a layer of spatially-varying refractive index above an unbounded rough surface on which the field vanishes. In particular, in the 2D case with TE polarization, the boundary value problem models the scattering of time harmonic electromagnetic waves by an inhomogeneous conducting or dielectric layer above a perfectly conducting unbounded rough surface, with the magnetic permeability a fixed positive constant in the medium. Via analysis of an equivalent variational formulation, we show that this problem is well-posed in two important cases when the frequency is small enough; and when the medium in the layer has some energy absorption. In this latter case we also establish exponential decay of the solution with depth in the layer. An attractive feature is that all constants in our estimates are bounded by explicit functions of the index of refraction and the geometry of the scatterer.

18. D. Colton and R. Kress, Using fundamental solutions in inverse scattering, *Inverse Problems* **22**, R49-R66 (2006).

    Abstract: We provide a brief description of recent results in inverse scattering theory having as a common mathematical framework the exploitation of the behaviour of the fundamental solution to the Helmholtz equation, in particular the fact that for the source point on the boundary ∂D of the scattering object such a solution is not in the Sobolev space $H^{1/2}(\partial D)$. Included in our discussion are uniqueness theorems, decomposition methods (including the point-source method), the method of singular sources, the linear sampling method and the factorization method.

19. D. Colton and P.B. Monk, Target identification of coated objects, *IEEE Trans. Antennas Prop.* **54**, 1232-1242 (2006)

    Abstract: We consider the three dimensional electromagnetic inverse scattering problem of determining information about a coated object from a knowledge of the electric far field patterns corresponding to time harmonic incident plane waves at fixed frequency. We assume that the obstacle is either a perfect conductor coated by a thin dielectric layer or a dielectric coated by a thin layer of a highly conducting material, i.e. the the coated portion of the boundary is modeled by either an impedance boundary condition or a conductive boundary condition. No a priori assumption is made on the connectivity of the scattering obstacle nor on the extent of the coating, i.e. the object can be either fully coated,partially coated or not coated at all. We present an algorithm based on the linear sampling method for reconstructing the shape of the scattering obstacle together with an estimate of either the surface impedance or surface conductivity. Numerous numerical examples are given showing the efficaciousness of our method.

20. D. Colton, L. Päivärinta and J. Sylvester, The interior transmission problem, *Inverse Problems and Imaging* **1**, 13-28 (2007).

Abstract: The interior transmission problem is a boundary value problem that plays a basic role in inverse scattering theory but unfortunately does not seem to be included in any existing theory in partial differential equations. This paper presents old and new results for the interior transmission problem, in particular its relation to inverse scattering theory and new results on the spectral theory associated with this class of boundary value problems.

21. E. Darrigrand and P.B. Monk, Coupling of the ultra-weak variational formulation and an integral representation using a fast multipole method in electromagnetism, *J. Comp. Appl. Math.* **204**, 400-407 (2007).

Abstract: Many different methods have been developed for the solution of the time-harmonic Maxwell equations in exterior domains at high frequency. Volume based methods have the drawback of needing an artificial boundary far from the obstacle. Integral formulations enable one to avoid this difficulty by solving a problem on the surface of the obstacle. However integral operators imply dense systems with bad condition numbers. The Ultra-Weak Variational Formulation (UWVF) is a volume based method using plane wave basis functions that allows the use of a coarser mesh in comparison with more classical low order finite element methods. However the UWVF still involves the problem of the artificial boundary. In this paper, we suggest the use of an integral representation of the unknown field to obtain an exact artificial boundary condition. In this way the distance between the obstacle and the artificial boundary can be reduced. The use of the Fast Multipole Method ensures a low cost for the calculation of various integral operators used in the representation. In this paper we describe the combined algorithm, demonstrate its accuracy on a model problem and discuss the complexity of the algorithm.

22. M. Malinen, P.B. Monk and T. Huttunen, Solving Maxwell's equations using the ultra weak variational formulation, *J. Comp. Physics* **223**, 731-758 (2007).

Abstract: We investigate the ultra weak variational formulation for simulating time-harmonic Maxwell problems. This study has two main goals. First, we introduce a novel derivation of the UWVF method which shows that the UWVF is an unusual version of the standard upwind discontinuous Galerkin (DG) method with a special choice of basis functions. Second, we discuss the practical implementation of an electromagnetic UWVF solver. In particular, we propose a method to avoid the conditioning problems that are known to hamper the use of the UWVF for problems in general geometries and inhomogeneous media. In addition, we show how to

implement the PML in the UWVF to accurately approximate physically unbounded problems and discuss the parallelization of the UWVF. Three dimensional numerical simulations are used to examine the feasibility of the UWVF for simulating wave propagation in inhomogeneous media and scattering from complex structures.

23. P.B. Monk and T. Huttunen, The use of plane waves to approximate wave propagation in anisotropic media, *J. Comp. Math.* **25**, 350-367 (2007).

Abstract: In this paper we extend the standard Ultra Weak Variational Formulation of Maxwell's equations in isotropic medium to the case of an anisotropic medium. We verify that the underlying theoretical framework carries over to anisotropic media (however error estimates are not yet available) and completely describe the new scheme. We then consider TM mode scattering, show how this results in a Helmholtz equation in two dimensions with an anisotropic coefficient and show how to formulate the UWVF for it. In one special case, convergence can be proved. We then show some numerical results that suggest that the UWVF can successfully simulate wave propagation in anisotropic media.